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LETTER TO THE EDITOR

Hamiltonian systems with a certain stability property

F T Hioe

Department of Physics, St John Fisher College, Rochester, NY 14618, USA

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Abstract. By requiring a Hamiltonian system with several adjustable parameters to have a certain stability property, a family is created which includes as its members the integrable cases. The results, obtained analytically, for two Hamiltonian systems with two degrees of freedom, are presented.

In this letter, I will present some analytic results using a new concept of stability which groups a given multiparameter Hamiltonian system possessing the same property into a 'family' that includes the integrable cases as its members.

The problem of integrability, local and global stability and instability, of a dynamical system has been the subject of intense study for many years [1]. For a Hamiltonian system with N degrees of freedom, integrability means the existence of N independent analytic global integrals of the motion. There have been several ingenious discoveries of integrable Hamiltonian systems through applications of inverse scattering transforms [2]. Of great importance and interest also is the suggestion that dynamical systems with the so-called Painlevé property are integrable [3-9].

Recently, there have been some remarkable results which came out from studies of the local stability or instability of a class of straight line periodic motions. On the one hand, Ziglin [10], Yoshida [11], Ito [12] and Yoshida *et al* [13] considered the monodromy properties around these particular solutions of a system and were able to prove some conditions for the non-integrability of a system, in the sense that no analytic integral of motion exists other than the Hamiltonian itself. On the other hand, Hioe and Deng [14] found a universal exponent by which the Lyapunov exponent approaches zero at the many or infinitely many stability-instability transition points for Hamiltonian systems of any dimensions. Furthermore, I have found the following result [15] which forms the basis of this letter. By requiring that any small perturbations to these straight line periodic solutions not only remain bounded, but also be *periodic* functions of the time not only for a specific initial value (a point) but for one or more specific *lines* of initial values, a family of systems has been found which includes the integrable cases as its members. Of even greater interest than the integrable cases themselves is what one can learn from other members (integrable or non-integrable) of the family.

In particular, I have applied this idea of requiring the system to have what I have called 'stability of type 1' property with respect to a line or lines of initial values for the special straight line periodic solutions, to the two following Hamiltonian systems with two degrees of freedom, the Hamiltonians of which are given by

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}(Ax^2 + By^2) + Dx^4 + Cx^2y^2 + Ey^4 \tag{1}$$

and

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}(Ax^2 + By^2) + Cx^2y - \tfrac{1}{3}Dy^3 \tag{2}$$

where (1) may be referred to as the generalised coupled quartic oscillator system [16] and (2) the generalised Hénon–Heiles system [17], and where the parameters $A, B, \ldots, E$ may assume any real values. The straight line periodic solutions exist in both cases and they are expressible in terms of the Jacobi elliptic functions for a line or lines of initial values which are typically of the form

$$(x(0), y(0), \dot{x}(0), \dot{y}(0)) = (x_0, 0, 0, 0) \text{ or } (0, y_0, 0, 0) \tag{3}$$

where x_0 and y_0 can assume any real values along the x and y axes respectively as long as the motions remain bounded. The behaviour of any small deviations Δx and Δy from the specified lines of initial values is governed by equations of the form

$$\mathrm{d}^2(\Delta u)/\mathrm{d}\tau^2 = \{Pk^2 \operatorname{sn}^2(\tau, k) - Q\}(\Delta u) \tag{4}$$

where Δu stands for Δx or Δy, τ is some rescaled time, P and Q depend on the parameters $A, B, \ldots, E$ appearing in the Hamiltonians, and k is the modulus of the elliptic function $\operatorname{sn} \tau$ and also depends on $A, B, \ldots, E$ as well as on the initial value x_0 or y_0. Equation (4) is of the form of the Lamé equation [18]. The requirement that the system has stability of type 1 with respect to a line of initial values implies that the solution for Δu must be a *periodic* Lamé function for any initial values on the specified line. This requirement places conditions on P and Q and thus on the parameters $A, B, \ldots, E$, and I have been able to obtain these conditions analytically. Cases which satisfy this requirement are presented in tables 1 and 2 for the generalised coupled quartic oscillator system (1), and for the generalised Hénon–Heiles system (2), respectively. The stability of the type-1 requirement for (1) is demanded with respect to two lines of initial values $(x_0, 0, 0, 0)$ and $(0, y_0, 0, 0)$ and for (2) with respect to one line of initial values $(0, y_0, 0, 0)$; the number of lines being the same as the number of periodic straight line solutions which exist for the two Hamiltonian systems.

Table 1. Cases of the generalised coupled quartic oscillator system (1), which have stability of type 1 and which are also integrable.

Case	$A:B$	$D:C:E$
I	1:1	1:1:1
II	1:1	1:3:1
III	1:4	1:6:16
IV	4:1	16:6:1

Table 2. Eight cases of the generalised Hénon–Heiles system (2), which have stability of type 1. Cases 1 and 6 are known to be integrable.

Case	A/B	C/D
1	$\frac{1}{16}$	$-\frac{1}{16}$
2	$\frac{1}{16}$	$-\frac{5}{16}$
3	$\frac{9}{16}$	$-\frac{5}{16}$
4	0	$-\frac{1}{2}$
5	1	$-\frac{1}{2}$
6	1	-1
7	1	$-\frac{5}{2}$
8	4	$-\frac{5}{2}$

All the four cases in table 1, and cases 1 and 6 in table 2, turn out to be known integrable cases [3–8, 19, Greene (private communication reported in [4, 5])]. The six other cases in table 2 are not known to be integrable or non-integrable. They clearly share an important characteristic with the known integrable cases, but they do not pass the Painlevé test. One of the six cases, case 5, was shown by Ito [12] to satisfy his condition for the system to have an entire integral which is functionally independent of the Hamiltonian, but he was not able to prove integrability or non-integrability. The precise status of these six cases, if it can be decided, is of great interest and has important implications.

I should point out that my stability of type-1 analysis missed out two integrable cases for reasons which I shall explain. One case is for the generalised coupled quartic oscillator system (1) when $A = B = 0$, $D:C:E = 1:3:8$, and the other is for the generalised Hénon–Heiles system (2) when $A/B =$ any number, $C/D = -\frac{1}{6}$. The integrability of these two cases was shown in [6] and Greene (private communication). For these two cases, the motions of the systems turn out to be unstable in the neighbourhood of the very lines of initial values which I have used for my analysis. This instability can occur even though the systems are integrable because the gradients of the first and second integrals become, for these cases, linearly dependent on the straight lines, and they are therefore beyond the reach of Poincaré's theorem [20, 21] on the non-existence of an exponentially unstable solution in an integrable Hamiltonian system with two degrees of freedom. Thus these two rather exceptional cases have been missed out in my analysis. It shows that the stability of type-1 analysis may not lead to all possible integrable cases, and I do not claim that it would. Nevertheless, aside from getting most of the integrable cases, the family of cases that it has created is interesting in its own right, from which, I believe, we can learn much.

Besides applying it analytically, the concept is open to an even wider application in numerical experiments to all kinds of Hamiltonian systems for which the equations corresponding to (4) for the systems may be much more complex. Since most systems are non-integrable and integrable systems are exceptions, I hope that this approach will help to find most of the possible integrable cases of practical interest. More importantly, I hope that a better understanding and application will be found in the family of systems which have such an easily identifiable and appealing property as that of the stability of type 1.

Hamiltonian systems are also known to give rise to chaotic motions. For the coupled quartic oscillator system, the chaos–order–chaos transitions as the coupling parameter is varied was clearly demonstrated by the numerical studies of Deng and Hioe [22]. Stability of type-1 analysis showed that an integrable case, or any member of the family having the stability of type-1 property, is typically the 'boundary' that separates cases in which the behaviour of small deviations from a given initial value is quasiperiodic and cases in which small deviations grow exponentially.

In summary, I have introduced a useful property which I have called stability of type 1. The concept is somewhat misleadingly simple, and yet proves to be quite powerful. It can be used to find integrable systems and, more importantly perhaps, to characterise systems and motions which are more general than being simply integrable. It would be worthwhile to see whether the same success can be achieved for non-Hamiltonian and driven systems.

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